Recovery of nonlinearly degraded sparse signals through rational optimization

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Recovery of Nonlinearly Degraded Sparse Signals through Rational Optimization

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***Abstract*—We show the benefit which can be drawn from recent global rational optimization methods for the minimization of a regularized criterion. The regularization term is a rational Geman-MacClure like potential, approximating the ℓ_0 norm and the fit term is a least-squares criterion suited to a wide class of nonlinear degradation models.**

I. Introduction

Over the last decade, much attention has been paid to inverse problems involving sparse signals. A popular approach consists in formulating such problems under a variational form where one minimizes the sum of a data fidelity term and a regularization term incorporating prior information. For sparse signals, the regularization term may involve the ℓ_0 norm, or an approximation of it [1]. This generally results in difficult optimization problems with many local minima and weak global convergence guaranties [2]–[5]. In this work, we consider rational optimization algorithms offering global optimality guaranties. In addition, our method allows us to address the challenging case of a nonlinear model [6]–[8].

II. Model and criterion

Consider a sparse vector with unknown nonnegative samples $\overline{\mathbf{x}} := (\overline{x}_1, \dots, \overline{x}_T)^\top$, only a few of which are nonzero. We aim at recovering it from measurements $\mathbf{y} := (y_1, \dots, y_T)^\top$ related to $\overline{\mathbf{x}}$ through a linear transformation (typically, a convolution) followed by some nonlinear effects:

$$\mathbf{y} = \boldsymbol{\phi}(\mathbf{H}\overline{\mathbf{x}}) + \mathbf{n} \; , \tag{1}$$

where $\mathbf{n} := (n_1, \dots, n_T)^\top$ is a realization of a random noise vector, and $\boldsymbol{\phi} : \mathbb{R}^T \to \mathbb{R}^T$ is a rational nonlinear function with components $[\boldsymbol{\phi}(\mathbf{u})]_k = \phi(u_k)$ depending on the k^{th} entry u_k only. $\mathbf{H} \in \mathbb{R}^{T\times T}$ is a given convolution matrix, which is assumed Toeplitz banded under suitable vanishing boundary conditions. To estimate $\overline{\mathbf{x}}$, we minimize a penalized criterion having the following form:

$$(\forall \mathbf{x} \in \mathbb{R}_+^T) \quad \mathcal{J}(\mathbf{x}) = \|\mathbf{y} - \boldsymbol{\phi}(\mathbf{H}\mathbf{x})\|^2 + \lambda \sum_{t=1}^{T} \frac{x_t}{\delta + x_t} \; , \tag{2}$$

where λ and δ are positive regularization and smoothing parameters. The last term is a Geman-McClure like potential as in [9]. We assume that an upper-bound B on the values $(\overline{x}_t)_{t=1}^T$ is available and the minimization is thus performed over a compact set defined and represented by $\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^T \mid x_t(B - x_t) \geq 0, t = 1, \dots, T\}$. Then, the optimization problem consists in finding $\mathcal{J}^\star := \inf_{\mathbf{x}\in\mathbf{K}} \mathcal{J}(\mathbf{x})$.

III. Rational minimization

Given $\mathcal{J}$ in (2), the previous minimization is a rational problem. The methodology in [10, 11] builds for different orders k a hierarchical sequence of semi-definite programming (SDP) relaxations $\mathcal{P}_k^\star$ for which the following optimality result holds: $\mathcal{P}_k^\star \uparrow \mathcal{J}^\star$ as $k \to +\infty$.

By using SPD solvers to solve $\mathcal{P}_k^\star$, one can hence theoretically obtain the global optimum [9]. Due to the maximum tractable size of state of the art SDP solvers, this approach is however limited to small/medium size problems having small degree, even when restricting the hierarchy to a finite and small order k. To overcome this difficulty, we exploit the problem structure in the sum of rational terms in (2). Using the sparse Toeplitz banded shape of $\mathbf{H}$, it can be noticed that:

$$\mathcal{J}(\mathbf{x}) = \sum_{t=1}^{T} \underbrace{\left[y_t - \phi\left(\sum_{i=1}^{L} h_i x_{t-i+1}\right)\right]^2}_{\text{depends on } x_k \text{ for } k \in J_t} + \underbrace{\lambda \frac{x_t}{\delta + x_t}}_{\text{depends on } x_t \text{ only}} \; ,$$

where $J_t = \{\min\{1, t-L-1\}, \dots, t\}$ and $J_{t+T} = \{t\}$ for any $t \in \{1, \dots, T\}$. These index subsets satisfy the so-called "Running Intersection Property" [12]. As a consequence, it is possible to introduce a much smaller SDP relaxation $\mathcal{P}_k^{\star\mathrm{s}}$ instead of $\mathcal{P}_k^\star$. The fundamental idea is that the SDP relaxations involve variables representing monomials in $(x_1, \dots, x_T)$. Using the above split form, many monomials can be discarded, the most striking case being when $\mathcal{J}$ is fully separable.

IV. Experiments

We have generated 100 Monte-Carlo realizations of vector $\overline{\mathbf{x}}$ containing $T = 200$ sparse samples, exactly 20 of which are nonzero. The nonzero sample values were randomly drawn in $[\frac{2}{3}; 1]$. We have generated $\mathbf{y}$ according to (1) with the nonlinearity $\phi(u_k) = \frac{u_k}{0.3+u_k}$ and with additive i.i.d. zero-mean Gaussian noise with standard deviation $\sigma = 0.15$. The banded Toeplitz matrix $\mathbf{H}$ has been set in accordance with two choices of FIR filters of length 3 (denoted $\mathbf{h}^{(a)}$ and $\mathbf{h}^{(b)}$). We considered the estimate $\mathbf{x}_3^{\star\mathrm{s}}$ given by the optimal point of the SDP relaxation $\mathcal{P}_3^{\star\mathrm{s}}$ of order $k = 3$.

For comparison, we have implemented a proximal gradient algorithm based on Iterative Hard Thresholding (IHT) [3] extended to the the nonlinear model. Also, we tested a convex relaxation based on a a linearized reconstruction with ℓ_1 penalization. The local optimization algorithms have been started with different initializations and Table I indicates the existence of local minima.

On Figure 1, we have plotted the value $\mathcal{P}_3^{\star\mathrm{s}}$ reached by the SDP relaxation (which is a lower bound on $\mathcal{J}^\star$), the objective value $\mathcal{J}(\mathbf{x}_3^{\star\mathrm{s}})$ and the objective value reached using IHT using two different initializations. Clearly, our method provides a point close to a global minimizer and is very useful in providing a good initialization point for local optimization algorithms.

Finally, the estimation error has been quantified by $\|\hat{\mathbf{x}} - \overline{\mathbf{x}}\|$ for a given estimate $\hat{\mathbf{x}}$. The average error and objective values are summarized in Table II.

TABLE I
FINAL VALUES OF THE OBJECTIVE $\mathcal{J}(\mathbf{x})$ FOR THE GRADIENT AND IHT LOCAL OPTIMIZATIONS (AVERAGE OVER 100 MONTE-CARLO REALIZATIONS)

Gradient minimization					
Filter param.	Initialization				
	$\mathbf{x}_3^{\star s}$	ℓ_1	$\mathbf{y}$	zero	$\overline{\mathbf{x}}$
$\mathbf{h}^{(a)}$	6.9219	15.136	31.338	16.041	7.0894
$\mathbf{h}^{(b)}$	6.7078	13.245	30.222	18.060	7.0894

IHT minimization					
Filter param.	Initialization				
	$\mathbf{x}_3^{\star s}$	ℓ_1	$\mathbf{y}$	zero	$\overline{\mathbf{x}}$
$\mathbf{h}^{(a)}$	6.6943	8.4078	8.4129	16.041	6.7628
$\mathbf{h}^{(b)}$	6.6292	8.3442	8.2598	14.664	6.7372

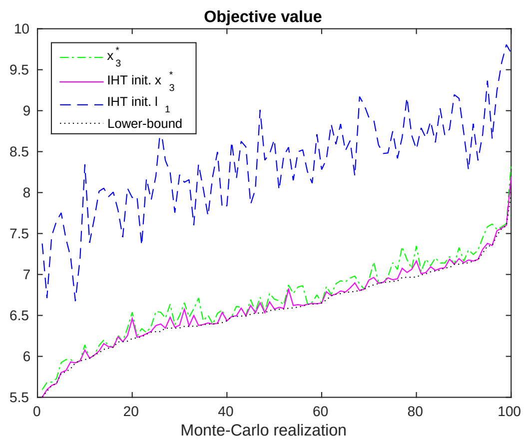

Fig. 1. Objective values provided by the different algorithms and lower-bound (using filter $\mathbf{h}^{(a)}$).

TABLE II
FINAL VALUES OF THE OBJECTIVE $\mathcal{J}(\mathbf{x})$ AND ESTIMATION ERROR GIVEN BY THE PROPOSED METHOD AND IHT WITH DIFFERENT INITIALIZATIONS (AVERAGE OVER 1000 MONTE-CARLO REALIZATIONS).

	Objective		Error	
Filter param.	$\mathbf{h}^{(a)}$	$\mathbf{h}^{(b)}$	$\mathbf{h}^{(a)}$	$\mathbf{h}^{(b)}$
Proposed method	6.9219	6.7078	1.3278	1.5408
Proposed method + IHT	6.6943	6.6292	1.3374	1.5393
linear + ℓ_1 +IHT	8.4078	8.3442	1.5575	1.6833

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